Note on Mr. Marth's "Intersects." By H. H. Turner, B.A., B.Sc

§ 1. In some recent numbers of the *Monthly Notices* (vol. xlv. pp. 370 and 483), Mr. Marth has described an interesting method of obtaining a graphical representation of the Solar System, each orbit being represented by its trace on a plane which passes through the centre of the Sun and revolves about an axis perpendicular to the ecliptic. The following brief remarks on the general nature of these curves may be of interest:—

§ 2. Let

S be the centre of the Sun, and P any point in the orbit.

i = inclination of orbit to the ecliptic.

ω = angle between line of nodes and major axis of orbit.

r = distance of P from S.

w = ,, ,, ,, ,, line of nodes.

x = ,, ,, ,, ,, directrix of orbit.

z = ,, ,, ,, ,, major axis of orbit.

y = ,, ,, ,, ,, ecliptic.

p = ,, ,, S ,, directrix of orbit.

Then we have the following relations among these quantities:—

$$\left.\begin{aligned} y &= w \sin i \\ r &= ex \\ r^2 &= (x-p)^2 + z^2 \\ w &= z \cos \omega - (x-p) \sin \omega \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

Eliminating w, x, z from these four equations, we get the equation

$$e^2 r^2 \cos^2 \omega = e^2 y^2 \operatorname{cosec}^2 i + 2ey(r-ep) \sin \omega \operatorname{cosec} i + (r-ep)^2 \quad . \quad . \quad (2)$$

and considering the point in the intersect corresponding to P, as defined by the coordinates r and y, this is the equation to the "intersect."

§ 3. If we write x in place of r in this equation (where x is now the usual Cartesian coordinate), we get the conic

$$x^2(1-e^2\cos^2\omega) + 2e \sin \omega \operatorname{cosec} i \, . \, xy + e^2 y^2 \operatorname{cosec}^2 i, \\ -2e^2 p \sin \omega \operatorname{cosec} i \, . \, y - 2ep\,x + e^2 p^2 = 0.$$

The second invariant is

$$e^2 \sin^2 \omega \operatorname{cosec}^2 i \, (e^2 - 1)$$

N 2

so that the conic is an ellipse, parabola, or hyperbola, as e is $<$, $=$, or $>$ 1; *i.e.* is of the same kind as the orbit.

§ 4. The "intersect" may be laid down graphically from this conic in the following manner:—Draw a series of lines parallel to the axis of x, and cutting the axis of y; and with radius equal to the distance on one of these lines between the axis of y and the curve, and the origin as centre, cut this line by a small circular arc. The points of intersection will trace out the "intersect."

The reverse operation will reproduce the conic from the intersect. I have shown in a diagram (Plate 4) some of these conics traced from Mr. Marth's intersects in *Monthly Notices*, vol. xlv. plate 3. Corresponding points in two curves are at the same distance from the axis of x; and one set of curves may be regarded as a kind of distortion of the other.

It is readily seen that

(a) The intersect as given by the equation is symmetrical to the axis of y, since r contains x^2 only. In actual fact, however, we limit the plane to one side of the axis of y.

(b) At points on the axis of x the conic and intersect touch each other, and are generally similar in the neighbourhood of this axis.

(c) Points on the lines $y = \pm x$ for the conic correspond to points on the axis of y for the intersect, and the intersect corresponding to portions of the conic in the quadrants made by these lines remote from the axis of x is imaginary.

In the particular case under consideration, the points of intersection of the conic with the lines $y = \pm x$ are given by the equation

$$[x(e \sin \omega \operatorname{cosec} i + 1) - ep]^2 + e^2 x^2 \cos^2 \omega \cot^2 i = o. \quad . \quad . \quad (3)$$

which gives imaginary values of x, unless either $\cot i = o$, in which case the orbit is perpendicular to the ecliptic, and the intersect coincides with it, the conic becoming (part of) a straight line parallel to the axis of y; or $\cos \omega = o$, in which case the intersect is part of a conic, and the conic again becomes part of a straight line. The equations of course represent the whole of this conic and this straight line respectively, but we must adduce further considerations to determine which parts of the curves actually correspond to the orbit.

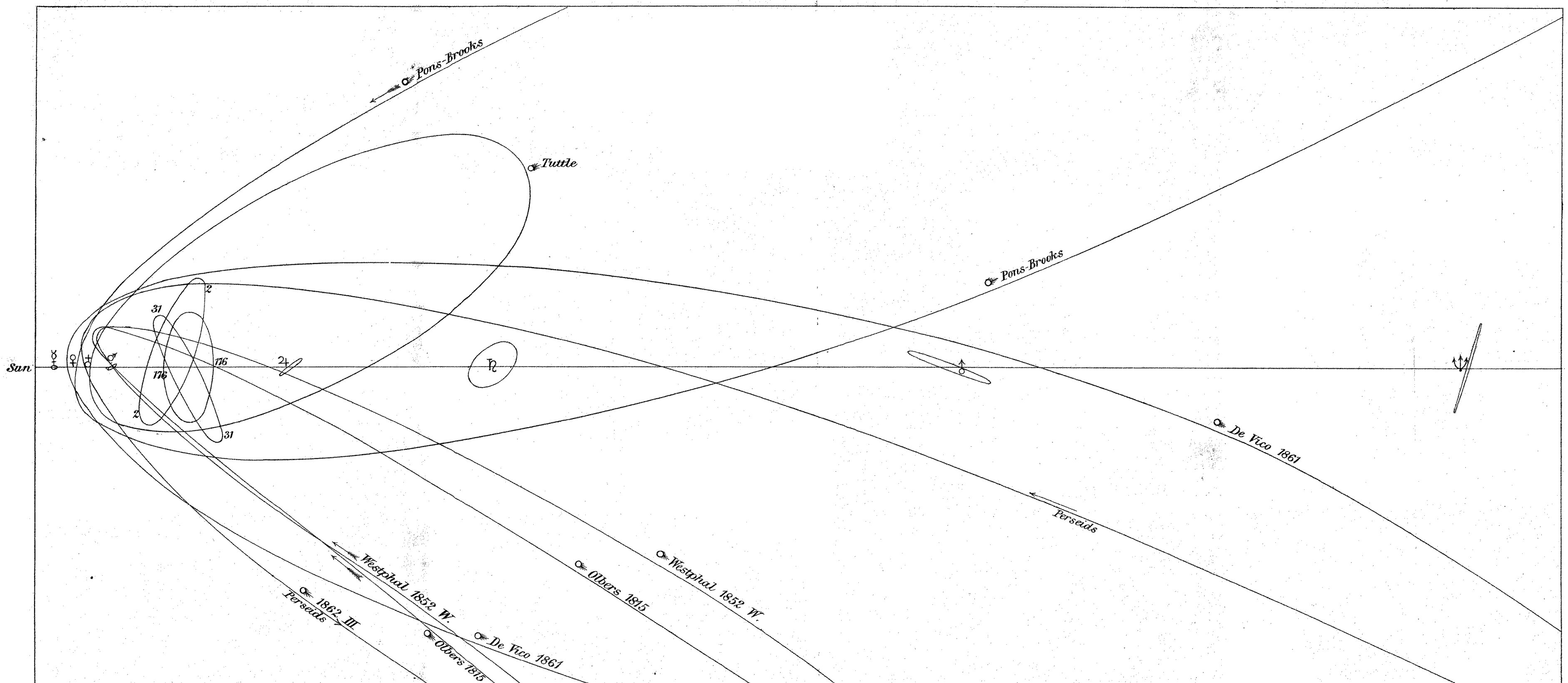
(d) If the conic breaks up into two straight lines, the "intersect" breaks up into two conics with axis of y as major axis and origin as focus. In fact, the equation to the intersect is then of the form

$$(ar + by + c)(a'r + b'y + c') = o,$$

and

$$r = -\frac{by}{a} - \frac{c}{a}$$

evidently represents such a conic.



Spottiswoode & Co. Lith. London.

§ 5. As regards the representation of the Solar system, there are several points in which a diagram of these conics would give much the same information as the diagram of intersects; and, for the construction of such a diagram, in place of the series of coordinates for different points tabulated so carefully by Mr. Marth, it would only be necessary to specify the coordinates of the centre, inclination and magnitude of the axes of the conic, to enable any one to reproduce the curves with an ellipsograph. Ambiguity as regards the position of the *major* axis might be removed by giving also the coordinates of the points where the conic cuts the axis of x.

The following expressions for these quantities are readily deducible. If a be the major axis of the orbit, putting

$$e = \sin\phi, \ \sin i \cos\omega = \cot\psi$$

coordinates of centre of conic are

$$x = a \qquad y = -a \sin\phi \sin i \sin\omega :$$

the inclination of the axis to the axis of x is θ, where

$$\tan 2\theta = \frac{\sin\phi \tan\omega \sin 2\psi}{\cos^2\psi \sec^2\omega - \sin^2\phi}$$

The axes are α and β, where

$$\frac{1}{\alpha} \pm \frac{1}{\beta} = \frac{2\sqrt{\cos^2\psi \tan^2\omega + \cos^2(\phi \mp \psi)}}{a \cos\psi \sin 2\phi}$$

§ 6. When the eccentricity of the orbit, and therefore of the corresponding conic, is nearly unity, as in the case of a comet, it may be advisable, owing to the centre occurring outside the limits of the paper employed for plotting the curves, to treat the conic as a parabola, and in this case the position of the focus and directrix will be more useful. The expressions for the coordinates of the focus when e is unity are

$$\alpha = \frac{q(1 + \operatorname{cosec}^2 i)}{2(\sin^2\omega + \operatorname{cosec}^2 i)} \text{ and } \beta = \alpha \,.\, \frac{\sin\omega}{\sin i}$$

and the equation to the directrix is

$$x + y \sin\omega \sin i = \frac{q}{2} \cos^2 i$$

where q is the semi-latus-rectum of the orbit. The conic can then be drawn by any of the mechanical methods for describing a parabola with given focus and directrix.

§ 7. A diagram of these conics would exhibit facts about perihelion distances perhaps more readily than that of the actual intersects: it need only be remembered that proximity to the Sun is here translated into proximity to the axis of y; and the perihelion and aphelion distances are thus distances from the axis of y to tangents parallel to it.

Facts about "inclination," on the other hand, are not exhibited so obviously. In the neighbourhood of the axis of x the inclination of the radius vector to the axis of x for the conic is nearly equal to that of the corresponding radius in the orbit to the plane of the ecliptic. But a radius in the orbit perpendicular to the ecliptic is represented in the case of the conic by a radius at an angle of 45° with the axis of x; and between these two extremes the inclination suffers a distortion or reduction increasing in value from zero to one half.

Still all comparative properties of two orbits are represented in very much the same way; *e.g.* the agreement or divergence of two orbits of the same comet, computed from different elements, would be exhibited equally well by the diagram of conics; and, since Mr. Marth seems to regard this possibility of recognising at a glance the agreement or divergence of two orbits as one of the chief uses of his diagrams, perhaps the greater facility in tabulation might recommend this rather more artificial but similarly useful method of diagrammatic representation in some cases.

§ 8. I have made a diagram (Plate 4) of the conics corresponding to some of Mr. Marth's intersects, given in the *Monthly Notices*, vol. xlv. Plate 3, by the actual process described in § 4, so that the scale is in a sense the same as in his diagram. The curves for the major planets, being in the neighbourhood of the axis of x, are almost precisely the same on either system.

In the following table are given the calculated values of the coordinates of centre, inclination of axis, and lengths of axes of the conics corresponding to the major planets and a few of the minor planets. I find that these elements may be computed for each planet in less than half an hour:—

Planet.	Coordinates of Centre.		Inclination of Major Axis.		Lengths of Axes.	
			°	′		
Mercury	0·387	−0·005	159	29	0·083	0·038
Venus	0·723	−0·000	5	25	0·042	0·003
Earth	1·000	+0·000	0	0	—	0·000
Mars	1·524	+0·004	18	51	0·009	0·006
Jupiter	5·203	+0·005	25	14	0·292	0·005
Saturn	9·555	+0·009	28	6	0·578	0·358
Uranus	19·218	−0·012	163	49	0·921	0·032
Neptune	30·111	+0·008	74	42	0·988	0·013
(2)	2·763	+0·294	70	20	1·634	0·393
(31)	3·145	−0·276	115	7	1·533	0·302
(84)	2·363	−0·019	166	5	0·566	0·359
(153)	3·951	−0·079	142	18	0·832	0·229
(176)	3·192	−0·001	89	47	1·207	0·525
(190)	3·947	+0·065	32	31	0·762	0·112